Approximating Oscillatory Integrals and Application

Ethan Y. Jaffe

If φ is a phase function, and $a \in S^m_{\rho,\delta}$ is a symbol, then we may define the oscillatory integral operator $I(a,\varphi)$ as a distribution by

$$\langle I(a,\varphi), u\rangle = \int e^{i\varphi(x,\theta)} a(x,\theta) u(x)\ dx d\theta,$$

where the (usually non-convergent) integral is defined by integration by parts may times using a suitable differential operator L such that $L^t(e^{i\varphi}) = e^{i\varphi}$ (see [1, Theorem 1.1] for details). The purpose of this note is to prove the following:

Proposition 1.1. *Let $\chi \in C_c^\infty(\mathbf{R}^n)$ be 1 at 0. Let φ be a phase function and $a \in S^m_{\rho,\delta}$ a symbol. Form the oscillatory integrals:*

$$I(a,\varphi) = \int e^{i\varphi(x,\theta)} a(x,\theta)\ d\theta$$

and

$$I_\varepsilon(a,\varphi) = \int e^{i\varphi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta)\ d\theta,$$

(where this last integral actually converges). Then $I_\varepsilon \to I$ weakly.

Proof. We need only show that if k is sufficiently large then

$$\int e^{i\varphi(x,\theta)} L^k(au(1-\chi_\varepsilon))\ d\theta \to 0$$

for every $u \in C_c^\infty$. Here, L is a differential operator of the form

$$L = \sum a_j \partial_{\theta_j} + b_j \partial_{x_j} + c,$$

where $a_j \in S^0_{1,0}$, $b_j, c \in S^{-1}_{1,0}$ and $L^t e^{i\varphi} = e^{i\varphi}$. Then

$$L^k = \sum a_{\alpha,\beta} \partial_\theta^\alpha \partial_x^\beta,$$

where $a_{\alpha,\beta} \in S^{|\alpha|-k}_{1,0}$. So,

$$L^k(au(1-\chi_\varepsilon)) = \sum a_{\alpha,\beta} \sum C\partial_\theta^{\alpha_1}\partial_x^{\beta_1} a \partial_x^{\beta_2} u \partial_\theta^{\alpha_2}(1-\chi_\varepsilon).$$

Here the inner sum is taken over $\alpha_1+\alpha_2=\alpha$, $\beta_1+\beta_2=\beta$. However, if we only use the terms with $\alpha_2 = 0$, then the sum is just $L^k(au)(1-\chi(\varepsilon\theta))$, which is uniformly bounded by an integrable function, in θ, and hence goes to 0. For the terms $\alpha_2 > 0$,

$$\partial_\theta^{\alpha_2}(1-\chi_\varepsilon) = \varepsilon^{|\alpha_2|}\partial_\theta^{\alpha_2}(\chi)(\varepsilon\theta).$$

Thus we may estimate on $\operatorname{supp} u$,

$$|L^k(au(1-\chi_\varepsilon))| \lesssim \sum (1+|\theta|)^{|\alpha|-k} \sum (1+|\theta|)^{m-\rho|\alpha_1|+\delta|\beta_2|}\varepsilon^{|\alpha_2|}.$$

An individual term looks like

$$(1+|\theta|)^{|\alpha|-k+m-\rho|\alpha_1|+\delta|\beta_2|}\varepsilon^{|\alpha_2|}.$$

If we integrate this from 0 to 1, the integral is obviously bounded by an integrable function, and so the integral goes to 0 thanks to the ε terms. Over $(1,\infty)$, we may replace the bounds of $(1+|\theta|)^n$ with $|\theta|^n$. We in fact need only to integrate up to to $\sup_\{ |x| \colon x \in \operatorname{supp} \chi_\varepsilon\}$, since the integrand is 0 outside this region. On this region, we have the bounds

$$\varepsilon^{k-|\alpha|-m+\rho|\alpha_1|-\delta|\beta_2|}\varepsilon^{|\alpha_2|}.$$

The exponent is bounded below by

$$(1-\delta)k + (\rho-1)|\alpha_1| - m,$$

which is positive for large enough k. So the integral goes to 0 with ε. □

An application of this proposition is as follows:

Proposition 1.2. *Suppose X is an open subset of $\mathbf{R}^n$, and $f \in C^\infty(X)$ with $\operatorname{Im} f \geq 0$, and $df \neq 0$ if $f(x) = 0$. For $k > 0$ we define*

$$(f(x)+i0)^{-k} = \lim_{\varepsilon\to 0^+}(f(x)+i\varepsilon)^{-k},$$

where the limit is taken in $\mathcal{D}'(X)$. Then

$$(f(x)+i0)^{-k} = C_k \int_0^\infty e^{if(x)\tau}\tau^{-1+k}\, d\tau,$$

where the integral is determined by taking a smooth cutoff $\chi(\tau)$ of 0, writing $1 = \chi + (1-\chi)$, noticing that the integral with χ converges, and the integral is $1-\chi$ is an oscillatory integral (if we extend $1-\chi = 0$ for $\tau < 0$).

Proof. Using a change of variables and the Gamma function, we see that

$$(f(x) + i\varepsilon)^{-k} = \int_0^\infty e^{i(f(x)+i\varepsilon)\tau}\tau^{-1+k}\ d\tau.$$

It is clear that

$$\int_0^\infty e^{i(f(x)+i\varepsilon)\tau}\tau^{-1+k}\chi(\tau)\ d\tau \to \int_0^\infty e^{if(x)\tau}\tau^{-1+k}\chi(\tau)\ d\tau.$$

To get the convergence of the other part of the integral, we will need a version of the proposition above, but with a small parameter. The proof follows by the same argument. Set

$$I = \int_0^\infty e^{if(x)}\tau^{-1+k}\varphi(\tau)d\tau,$$

where $\varphi = (1 - \chi)$ for $\tau \geq 0$ and 0 otherwise, and

$$I_\varepsilon = \int_0^\infty e^{if((x)+i\varepsilon)}\tau^{-1+k}\varphi(\tau)d\tau.$$

We need to show that $I_\varepsilon \to I$. Fix $\delta > 0$ and consider the distribution

$$J_\delta = \int_0^\infty e^{if(x)}\tau^{-1+k}\varphi(\tau)\psi(\delta\tau)\ d\tau,$$

where $\psi(0) = 1$. Also set

$$I_{\varepsilon,\delta} = \int_0^\infty e^{i(f(x)+i\varepsilon)}\tau^{-1+k}\varphi(\tau)\psi(\delta\tau)\ d\tau.$$

Then

$$I_\varepsilon - I = (I_\varepsilon - I_{\varepsilon,\delta}) + (I_{\varepsilon,\delta} - J_\delta) + (J_\delta - I).$$

We need to show that when testing against any fixed u, then is ε and δ is small enough, then the right-hand side is small. By the proposition, we may always choose δ independently of ε to make the last term as small as we like. Treating the symbol in I_ε as $e^{-\varepsilon\tau}\tau^{-1+k}\varphi(\tau)$, and observing that the first factor and all its dervatives are uniformly bounded as $\varepsilon \to 0$, we may use the proposition (with a parameter) to show that we can choose δ small independent of ε to make the first term as small as we like. Since the integrals of both terms in the middle actually converge, and δ has been fixed not depending on ε, if ε is small enough, it is easy to make the middle term small, as well. □

In turn, this latter proposition is essential in examing $\mathrm{WF}((f(x)+i\varepsilon)^{-k})$ and $\text{sing supp}(f(x)+i\varepsilon)^{-k})$ in terms of generalities of oscillatory intergals. See [1, Exercise 7.6].

References

[1] A. Grigis and J. Sjörstrand, *Microlocal Analysis for Differential Operators*. Cambridge University Press, Cambridge, 1994.