Fourier Transform of Homogeneous Radial Distributions

Ethan Y. Jaffe

The purpose of this note is to prove the following theorem, and then its one-dimensional analogue:

Theorem 1. *Let $d \geq 2$, and consider the function on $\mathbf{R}^d$ defined by $|x|^a$ for $-d < a < 0$. Then $|x|^a$ exists as a tempered distribution and has Fourier transform $C_{d,a}|\xi|^{-d-a}$, where*

$$C_{d,a} = (2\pi)^{d/2} 2^{a+d/2} \Gamma\left(\frac{a+d}{2}\right)\left(\Gamma\left(-\frac{a}{2}\right)\right)^{-1} > 0.$$

[1]

We will use the following convention on the Fourier transform:

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\ dx.$$

Proof. Since $-d < a$, $|x|^a \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ and hence is a tempered distribution. We now develop part of the theory of homogeneous radial distributions.

Definition 2. We will call a distribution $u \in \mathcal{D}'(\mathbf{R}^d \setminus \{0\})$ (resp. $u \in \mathcal{S}'(\mathbf{R}^d)$)) homogeneous of order a if for all $\varphi \in C_c^\infty(\mathbf{R}^d \setminus \{0\})$ (resp. $\varphi \in \mathcal{S}(\mathbf{R}^d)$))

$$\langle u, \varphi \rangle = t^{d+a} \langle u, \varphi_t \rangle,$$

where $\varphi_t(x) = \varphi(tx)$. If $u \in \mathcal{D}'(\mathbf{R}^d)$ and the above is true for all $\varphi \in C_c^\infty(\mathbf{R}^d)$, then we say that u is homogeneous or order a on $\mathbf{R}^d$.

This definiton is set up so that homogeneous functions of order a are homogeneous distributinos of the same order. Differentiating with respect to t (and verifying the the formal computation is valid) yields that this is equivalent to the Euler homogeneity relation (c.f. [1, §3.2]):

$$(a+d)\langle u, \varphi \rangle + \langle u, r\partial_r \varphi \rangle = 0.$$

Here $r\partial_r = \sum x_j \partial_{x_j} = x \cdot \nabla$ is the radial vector field. Observing that the formal adjoint of $r\partial_r$ is $-r\partial_r - d$, it follows that the previous display is equivalent to

$$r\partial_r u = au. \tag{1}$$

[1] Please see my other note where I consider the same theorem for all a. By meromorphic continuous, this holds for all $a \notin \mathbf{Z}$ relatively easily. The other note is more complicated since the cases $a \in \mathbf{Z}$ involve significant difficulty.

Definition 3. We will call a distribution $u \in \mathcal{D}'(\mathbf{R}^d \setminus \{0\})$ radial if

$$\langle u, \varphi \rangle = \langle u, \varphi \circ T \rangle,$$

whenever $T \in SO(d)$ is a rotation.

We now give a useful property of radial distributions.

Lemma 4. *If $u \in \mathcal{D}'(\mathbf{R}^d \setminus \{0\})$ is radial then $Lu = 0$ for every vector field L such that L_x is tangent the sphere of radius $|x|$.*

Proof. Let η_ε be a system of radial mollifiers. Then $u * \eta_\varepsilon$ is radial as a distribution. Indeed, one checks that and so for any $T \in SO(d)$,

$$(\varphi \circ T) * (\check{\eta}_\varepsilon \circ T) = (\varphi * \check{\eta}_\varepsilon) \circ T.$$

Thus,

$$\langle u, \varphi \circ T * \check{\eta}_\varepsilon \rangle = \langle u, (\varphi * \check{\eta}_\varepsilon) \circ T \rangle = \langle u, \varphi * \check{\eta}_\varepsilon \rangle.$$

So $u * \eta_\varepsilon$ is radial. Changing variables then gives that $\int ((u * \eta_\varepsilon) \circ T^{-1})\varphi = \int u * \eta_\varepsilon \varphi$, and thus $u * \eta_\varepsilon$ is radial as a function. In particular $L(u * \eta_\varepsilon) = 0$ for any tangent vector field L. Taking $\varepsilon \to 0$ proves the lemma. □

We now state a proposition involving the Fourier transforms of homogeneous and radial distributions.

Proposition 5. *Suppose $u \in \mathcal{S}'(\mathbf{R}^d)$ is homogeneous of order a, then $\hat{u}$. homogeneous of order $-d-a$. Similarly, if u, considered as a distsribution in $\mathcal{D}'(\mathbf{R}^d \setminus \{0\})$ is radial, then so is $\hat{u}$.*

Proof. One only needs to know that for $\varphi \in \mathcal{S}'(\mathbf{R}^d)$, $\hat{\varphi}_t(\xi) = t^{-d}\hat{\varphi}_{1/t}(\xi)$ and $\widehat{\varphi \circ T} = \hat{\varphi} \circ T$ for $T \in SO(n)$. Then the proof is just an exercise in definitions. □

We can now show that the Fourier transform of $|x|^a$ is $C_{d,a}|\xi|^{-d-a}$, for some constant $C_{d,a}$. $|x|^a$ is homogeneous of order a and radial, and thus its Fourier transform is homogeneous of order $b = -d-a$ and radial. We show that the only such functions are multiples of $|x|^b$, which will show the first part of the theorem. Let $u \in \mathcal{S}'(\mathbf{R}^d)$ be a radial distribution which is homogeneous of order b when considered as a distribution on $\mathbf{R}^d \setminus \{0\}$. Then $v = |x|^{-b}u$ is radial and homogeneous of order 0 (notice that $|x|^{-b}$ is smooth on $\mathbf{R}^d \setminus \{0\}$). Thus by (1) $\partial_r v = 0$. However, if L is any vector field tangent to spheres, then $Lv = 0$ too. Since any vetor field on $\mathbf{R}^d \setminus \{0\}$ decomposes into a multiple of ∂_r and a vector field tangent to all circles, we deduce that $Lv = 0$ for any vector field V. It follows immediately that v is constant, i.e. $u = C|x|^b$, at least on $\mathbf{R} \setminus \{0\}$. In other words, $\text{supp}(u - C|x|^b) \subseteq \{0\}$. Thus $u - C|x|^b$ is a sum of δ functions and their derivatives. But such a function, if it is to be homogeneous, is homogeneous of order at most $-d$, or else is 0. Thus $u - C|x|^b \equiv 0$. Indeed, the Fourier transform of any sums of δ and its derivatives is a polynomial. If it is to be homogeneous, it is either 0 or a homogeneous polynomial of degree at least 0, i.e. the sum of δ themselves is either 0 or homogeneous or order at most $-d$, by the above proposition.

Next we determine $C_{a,d}$. Let $G(x) = e^{-|x|^2/2}$ be a standard Gaussian. Then $\hat{G}(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}$. Thus,

$$\int_{\mathbf{R}^d} |x|^a e^{-|x|^2/2} \, dx = C_{d,a} (2\pi)^{-d/2} \int_{\mathbf{R}^d} |\xi|^{-d-a} e^{-|x|^2/2} \, d\xi.$$

The left-hand side is, by a change of variables,

$$\omega_{d-1} \int_0^\infty r^{a+d-1} e^{-r^2/2} \, dr = \omega_{d-1} 2^{(a+d)/2-1} \Gamma\left(\frac{a+d}{2}\right),$$

where ω_{d-1} denotes the surface area of the unit $d-1$ sphere. Similarly, the right-hand side is

$$C_{d,a} \omega_{d-1} (2\pi)^{-d/2} 2^{-a/2-1} \Gamma\left(-\frac{a}{2}\right).$$

Thus,

$$C_{d,a} = (2\pi)^{d/2} 2^{a+d/2} \Gamma\left(\frac{a+d}{2}\right) \left(\Gamma\left(-\frac{a}{2}\right)\right)^{-1} > 0,$$

where the positivity follows since both arguments of the Γ function are real and positive. If a is not a negative integer, we can write this in another way. Observe that since $\Gamma(t+1) = t\Gamma(t)$,

$$\Gamma\left(\frac{a+d}{2}\right) = 2^{-d-1} \left(\prod_{k=1}^{d-1} (a+d-k)\right) \Gamma(a/2 + 1/2).$$

Next, observe that by duplication,

$$\Gamma(a/2 + 1/2) = 2^{-a} \sqrt{\pi} \frac{\Gamma(a+1)}{\Gamma(a/2+1)}.$$

Finally, notice that by the reflection formula

$$\Gamma(a/2+1)\Gamma(-a/2) = -\frac{\pi}{\sin(\pi a/2)}.$$

Putting it all together,

$$C_{d,a} = -2\pi^{(d-1)/2} \sin(\pi a/2) \Gamma(a+1) \prod_{k=1}^{d-1} (a+d-k).$$

□

Now we turn to the one-dimensional case. In one dimension, $\mathbf{R} \backslash \{0\}$ is not connected, and the notion of a "radial" distribution does not make sense. However, there are no directions tangent to a sphere. So we can follow the proof above and deduce that if u is a homogeneous distribution of order a for $-1 < a < 0$, then u is a multiple of $|x|^a$, perhaps a different multiple on the positive and negative axes. If furthermore u is a real-valued even distribution (i.e. one which restricts to a distribution for real-valued functions and is even in the obvious sense), then so is its Fourier transform. We deduce that if $u = |x|^a$, then $\hat{u} = C_{1,a}|\xi|^{-1-a}$, where we can compute $C_{1,a}$ as above (the product is just empty). One can also compute the Fourier transform of distributions such that $v = \chi_{x>0}|x|^a = x_+^a$, which is aso homogeneous or order a. Indeed, $v(x)+v(-x) = |x|^a$, and $v(x)-v(-x) = |x|^a \operatorname{sgn}(x)$. The Fourier transform of the former is $\hat{v}(\xi) + \hat{v}(-\xi) = C_{1,a}|x|^a$. Since $|x|^a \operatorname{sgn}(x)$ is odd and real-valued, so is its Fourier transform. Thus its Fourier transform is of the form $C'_a|\xi|^{-1-a} \operatorname{sgn}(\xi)$ for some constant C'_a. One can compute C'_a in a similar way to computing $C_{d,a}$, except one uses $H(x) = xe^{-x^2/2}$ instead of $G(x) = e^{-x^2/2}$. Thus, $\hat{v}(\xi) - \hat{v}(-\xi) = C'_a|\xi|^{-1-a} \operatorname{sgn}(\xi)$. This gives a system of 2 equations with two unknowns $\hat{v}(\xi)$ and $\hat{v}(-\xi)$, which can then be solved for. We leave it as an exercise to the reader to determine exactly what C'_a, and $\hat{v}(\xi)$ are.

References

[1] Lars Hörmander, *The Analysis of Linear Partial Differential Operators I.* Springer, Berlin, 1990.